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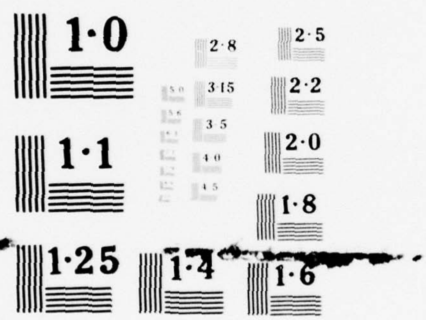

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GALERKIN-TYPE APPROXIMATIONS WHICH ARE DISCONTINUOUS IN TIME FOR PARABOLIC EQUATIONS IN A VARIABLE DOMAIN

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ABSTRACT

We consider general linear parabolic equations in a given time dependent domain and we describe a general class of Galerkin-type approximations which are continuous with respect to the space variables, but which admit discontinuities with respect to time at each time step. Unconditional stability is proved and a general error estimate is established. These results are applied to certain finite element methods based on space-time finite elements.

AMS(MOS) Subject Classification - 65N05

Key Words - Galerkin-type method, Finite elements, Parabolic equations, Moving boundary, Stability, Error estimate.

Work Unit Number 7 - Numerical analysis

EXPLANATION

Most numerical methods for parabolic equations are based on a space discretization which is independent of time and are not appropriate for problems in a variable domain. These problems can be approximated by using finite elements which are relative to both the space and time variables; such elements have been used for free boundary problems in heat conduction (Stefan problem) and fluid flow, but no mathematical results have been proved. In this report, we consider the case of a given moving boundary and we prove stability and convergence with error estimates for a general class of Galerkin-type methods which includes the case of space-time finite elements.

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GALERKIN-TYPE APPROXIMATIONS WHICH ARE DISCONTINUOUS IN TIME FOR PARABOLIC EQUATIONS IN A VARIABLE DOMAIN

Pierre Jamet*

1. Introduction

Most numerical methods for parabolic partial differential equations are based on a space discretization which is independent of time; this is the case for most finite element or Galerkin-type methods which have been studied in recent years ([2], [6], [7], [21], [23], [24], [26], [27]). These methods are not appropriate for parabolic problems in time-dependent domains, in particular for time-dependent free boundary problems.

In order to solve such problems, numerical methods based on space-time finite elements, i.e. on finite elements which are relative to both the space and time variables have been proposed and numerically tested ([3], [4], [11], [12]). Let us notice that space-time finite elements have been first considered by J. T. Oden [19](**), but only in the case when the elements are the Cartesian product of a space element by a time interval, which yields a space discretization which remains fixed in time.

Another finite element method to deal with variable domain has been studied by M. Mori [16], [17] for the Stefan problem; it is based on finite elements in space which depend continuously on time. This method is indeed a particular case of the generalized Galerkin method studied by Mignot [15]; in the same paper, Mignot studies other methods for parabolic equations in a variable domain: method of fixed auxiliary domain, method of elliptic regularization. Let us finally mention the finite difference methods studied by the author [9] for parabolic equations of order 2 which can degenerate or admit singularities at the initial time as well as on the boundary of the variable domain.

In the present paper, we consider general parabolic equations in a <u>given</u> time-dependent domain and we present a general class of Galerkin-type methods with applications to space-time finite elements. These methods are different from those which have been

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studied previously (including the space-time finite element methods of [3], [4], [11], [12]): the approximations are continuous with respect to the space variables for each fixed time, but they admit discontinuities with respect to the time variable at each time step; in particular, the elements can be chosen arbitrarily at each time step with no connection with the elements corresponding to the previous step.

For more simplicity, we first consider a model problem: the Dirichlet problem for the heat equation. This problem is presented in Section 2; we derive an integral relation which is satisfied by the solution and which is the basis of the numerical method. In Section 3, we describe a general class of time-discontinuous Galerkin-type approximations and prove unconditional stability; the discontinuities are treated as in the discontinuous finite-element methods of Reed and Hill [22] and Lesaint and Raviart [13] for the stationary neutron transport equation; the same technique has also been used by Pini [20]. In Section 4, we establish a general error estimate. In Section 5, we apply the previous results to space-time finite element methods; the order of accuracy can be made arbitrarily high by choosing finite elements of corresponding order. In Section 6, we give a simple example in the one-dimensional case; we give the explicit form of the discrete equations for rectangular space-time finite elements of order 1, which yields a finite difference analogue of our method. Finally, in Section 7, we extend the method and the previous results to general parabolic equations of order ≥ 2 with general boundary conditions.

2. A model problem

Let us consider a time interval $0 \le t \le T$ and let $\Omega(t)$ be a bounded domain in $\mathbb{R}^m$, m positive integer, which depends continuously on $t \in [0,T]$. Let $\Gamma(t)$ be the boundary of $\Omega(t)$ and

$$\mathcal{D}_T = \{(x,t);\ x \in \Omega(t),\ \ 0 < t < T\},$$

$$\Sigma_T = \{(x,t);\ x \in \Gamma(t),\ \ 0 < t < T\}.$$

$\mathcal{D}_T$ is a $(m+1)$-dimensional domain in $\mathbb{R}^m \times \mathbb{R}^+$ and Σ_T is its "lateral" boundary. We assume that Σ_T is piecewise smooth. For simplicity, we will use the same notation $\Omega(t)$ for the domain $\Omega(t) \subset \mathbb{R}^m$ and for the corresponding section of $\mathcal{D}_T$, i.e. the set $\{(x,t);\ x \in \Omega(t),\ t \text{ fixed}\}$.

Let f and u^0 be two given functions, $f \in L^2(\mathcal{D}_T)$, $u^0 \in L^2(\Omega(0))$, and let Δ denote the Laplacian operator with respect to the space variables. We consider the problem

$$(2.1)\qquad \begin{aligned} &\text{a)}\quad \frac{\partial u}{\partial t} - \Delta u = f && \text{in } \mathcal{D}_T, \\ &\text{b)}\quad u = 0 && \text{on } \Sigma_T, \\ &\text{c)}\quad u = u^0 && \text{in } \Omega(0). \end{aligned}$$

For the existence and uniqueness of a classical solution u, under suitable hypotheses on f and u^0, see [8], [9]; for the existence and uniqueness of a weak solution, see [14], [15]. In this paper, we will assume that the solution u is sufficiently regular for the validity of the error estimates.

We will use the following notation:

$d = \mathrm{Max}\{\mathrm{diam}\ \Omega(t);\ \ 0 \le t \le T\}$,

$(\cdot,\cdot)_{\Omega(t)}$ = inner product in $L^2(\Omega(t))$,

$|\cdot|_{\Omega(t)}$ = norm in $L^2(\Omega(t))$,

$((\cdot,\cdot))_G$ = inner product in $L^2(G)$, where G is a subdomain of $\mathcal{D}_T$,

$\|\cdot\|_G$ = norm in $L^2(G)$.

The same notation will be used for vector valued functions in $(L^2(G))^m$. Thus,

$$((\mathrm{grad}\ \psi,\ \mathrm{grad}\ \varphi))_G = \iint_G \mathrm{grad}\ \psi \cdot \mathrm{grad}\ \varphi\ dxdt,$$

where ψ and φ are two arbitrary smooth enough functions.

Now, let τ_0 and τ_1 be two arbitrary numbers such that $0 \leq \tau_0 < \tau_1 \leq T$ and let $G = G(\tau_0, \tau_1)$ denote the intersection of $\mathcal{J}_T$ with the strip $\tau_0 < t < \tau_1$, i.e. $G = \{(x,t);\ x \in \Omega(t),\ \tau_0 < t < \tau_1\}$. Consider the bilinear form

$$(2.2) \quad \mathcal{B}_G(\psi,\varphi) = -((\psi, \frac{\partial\varphi}{\partial t}))_G + ((\text{grad } \psi, \text{ grad } \varphi))_G + (\psi,\varphi)_{\Omega(\tau_1)} - (\psi,\varphi)_{\Omega(\tau_0)} .$$

Let $\Phi(G)$ be *the space of all Lipschitz-continuous functions* φ defined on $\overline{G}$ (closure of G) and which vanish on the lateral boundary of G, i.e. on $\Sigma_T \cap \overline{G}$. Then, a classical integration by parts shows that the solution u of problem (2.1) satisfies the integral relation

$$(2.3) \quad \mathcal{B}_G(u,\varphi) = ((f,\varphi))_G \quad \text{for all } \varphi \in \Phi(G) \text{ and for all } G = G(\tau_0,\tau_1)$$
$$\text{with } 0 \leq \tau_0 < \tau_1 \leq T .^{(*)}$$

This relation is the basis of the numerical method described in the following section.

(*) It is possible to relax the hypotheses on the function φ in relation with the regularity of u. This question is considered in Section 6 where a precise variational formulation is given for more general parabolic problems.

3. Discontinuous approximations

Let $\{t^n;\ 0 \le n \le N\}$ be a finite sequence of real numbers with $t^0 = 0$, $t^n < t^{n+1}$ and $t^N = T$. Let $\Omega^n = \Omega(t^n)$, $G^n = G(t^n, t^{n+1})$ and $\tilde{G}^n = \bar{G}^n - \bar{\Omega}^n = \{(x,t);\ x \in \bar{\Omega}(t),\ t^n < t \le t^{n+1}\}$ (see Figure 1).

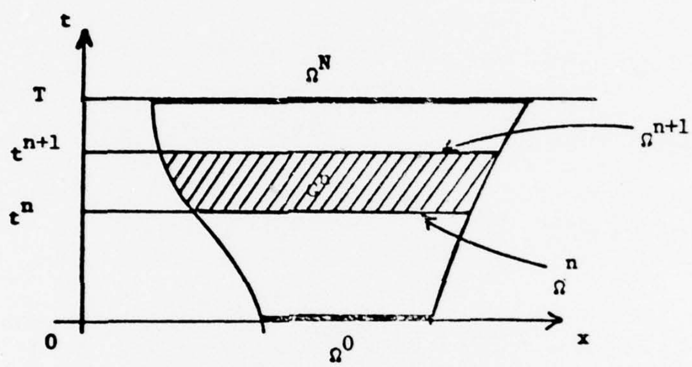

Figure 1: The discretization of the domain $\mathcal{D}_T$ with respect to time (one-dimensional case)

Let Φ_h^n be a finite dimensional subspace of $\Phi(G^n)$, for $0 \le n \le N-1$, and let V_h be the space of all functions v_h defined on $\bar{\mathcal{D}}_T$ such that their restriction to each $\tilde{G}^n$ coincides with the restriction to $\tilde{G}^n$ of a function $\varphi_h \in \Phi_h^n$. The functions $v_h \in V_h$ are in general discontinuous at the time $t = t^n$; we will use the notation $v_h^n = v_h(\cdot, t^n)$ for $0 \le n \le N$ and $v_h^{n+0} = \lim\{v_h(\cdot, t^n+\varepsilon);\ \varepsilon > 0,\ \varepsilon \to 0\}$ for $0 \le n \le N-1$; it follows from the definition of the space V_h that we have $v_h^n = \lim\{v_h(\cdot, t^n-\varepsilon);\ \varepsilon > 0,\ \varepsilon \to 0\}$ for $1 \le n \le N$.

We approximate problem (2.1) by the following problem which is a discrete analogue of (2.3). Discrete problem: Find $u_h \in V_h$ such that $u_h^0 = u^0$ and

(3.1) $$\mathcal{B}_{G^n}(u_h, \varphi_h) = ((f, \varphi_h))_{G^n},$$

for all $\varphi_h \in \Phi_h^n$ and for all n, $0 \le n \le N-1$.

We will prove the existence and uniqueness of the solution u_h of Problem (3.1); for that purpose we need two preliminary lemmas.

For each n, $0 \leq n \leq N-1$, there corresponds to each $v_h \in V_h$ a unique function $\varphi_h = q^{(n)} v_h \in \Phi_h^n$ which coincides with v_h on $\bar{G}^n$. Let

$$\mathcal{B}^n(u_h, v_h) = \mathcal{B}_{G^n}(u_h, q^{(n)} v_h), \quad \text{for all } v_h \in V_h \ . \tag{3.2}$$

$\mathcal{B}^n(u_h, v_h)$ is a bilinear form defined on $V_h \times V_h$ and we have

<u>Lemma 3.1</u>

For all n, $0 \leq n \leq N-1$, for all $u_h \in V_h$ and all $v_h \in V_h$,

$$\mathcal{B}^n(u_h, v_h) = -((u_h, \frac{\partial}{\partial t} v_h))_{G^n} + ((\text{grad } u_h, \text{ grad } v_h))_{G^n} + \tag{3.3}$$

$$+ (u_h^{n+1}, v_h^{n+1})_{\Omega^{n+1}} - (u_h^n, v_h^{n+0})_{\Omega^n} \ ,$$

$$\mathcal{B}^n(v_h, v_h) = \| \text{grad } v_h \|^2_{G^n} + \frac{1}{2} | v_h^{n+1} |^2_{\Omega^{n+1}} - \frac{1}{2} | v_h^n |^2_{\Omega^n} + \frac{1}{2} | v_h^{n+0} - v_h^n |^2_{\Omega^n} \ . \tag{3.4}$$

<u>Proof</u>: For each v_h, the function $\varphi_h = q^{(n)} v_h$ is continuous on $\bar{G}^n$, hence $\varphi_h(\cdot, t^n) = v_h^{n+0}$ and formula (3.3) follows at once from (2.2). Then, we deduce (3.4) by taking $v_h = u_h$, integrating the first term in the right hand side of (3.3) with respect to t and using the identity

$$(v_h^n, v_h^{n+0})_{\Omega^n} = \frac{1}{2} | v_h^n |^2_{\Omega^n} + \frac{1}{2} | v_h^{n+0} |^2_{\Omega^n} - \frac{1}{2} | v_h^{n+0} - v_h^n |^2_{\Omega^n} \ . \qquad \blacksquare$$

We will also need the following elementary lemma.

<u>Lemma 3.2</u>

Let a, b, c be three nonnegative real numbers such that $a^2 \leq b^2 + ac$. Then, $a \leq b + c$.

<u>Proof</u>: We have $a \leq \frac{1}{2}(c + (c^2 + 4b^2)^{1/2})$ and $(c^2 + 4b^2)^{1/2} \leq c + 2b$. $\blacksquare$

<u>Theorem 3.1</u>

<u>The discrete problem</u> (3.1) <u>admits a unique solution</u> u_h <u>which satisfies the estimate</u>

(3.5) $$\|\operatorname{grad} u_h\|^2_{G(0,t^n)} + \frac{1}{2}|u_h^n|^2_{\Omega^n} \le \frac{1}{2}(|u^0|_{\Omega^0} + \sqrt{2}\, d\, \|f\|_{G(0,t^n)})^2 ,$$

for all n, $0 \le n \le N$, with $d = \underset{0 \le t \le T}{\text{Max}} \{\operatorname{diam} \Omega(t)\}$.

Proof: For each n, (3.1) is a system of linear algebraic equations with a square matrix whose order is equal to the dimension of the space Φ_h^n . Hence, the existence of the solution u_h will be a consequence of its uniqueness.

Taking $\varphi_h = q^{(n)} u_h$ in (3.1), using (3.4), summing with respect to n and applying Poincaré's inequality, we get

$$\|\operatorname{grad} u_h\|^2_{G(0,t^n)} + \frac{1}{2}|u_h^n|^2_{\Omega^n} \le$$

$$\le \frac{1}{2}|u^0|^2_{\Omega^0} + \|f\|_{G(0,t^n)} \|u_h\|_{G(0,t^n)} \le$$

$$\le \frac{1}{2}|u^0|^2_{\Omega^0} + d\,\|f\|_{G(0,t^n)} \|\operatorname{grad} u_h\|_{G(0,t^n)} .$$

The estimate (3.5) follows at once by application of Lemma 3.2 with

$$a = (\|\operatorname{grad} u_h\|^2_{G(0,t^n)} + \frac{1}{2}|u_h^n|^2_{\Omega^n})^{1/2} ,$$

$$b = \frac{1}{\sqrt{2}}|u^0|_{\Omega^0} \quad \text{and} \quad c = d\,\|f\|_{G(0,t^n)} .$$

The uniqueness of u_h is a consequence of this estimate. ∎

Remark 3.1: The equations (3.1) are equivalent to

(3.6) $$\mathcal{B}(u_h, v_h) = ((f, v_h))_{\mathcal{J}_T} \quad \text{for all } v_h \in V_h ,$$

with $\mathcal{B}(u_h, v_h) = \sum_{n=0}^{N-1} \mathcal{B}^n(u_h, v_h)$. In (3.6) the index n does not appear. However, the form (3.1) is more convenient.

Remark 3.2. For each n, the space Φ_h^n can be chosen arbitrarily with no relation with the spaces Φ_h^ν, for $\nu \ne n$; in particular, the spaces Φ_h^n need not have the same dimension for different values of n .

Let us also notice that unlike other methods, the present method does not require a **preliminary** approximation of the initial function u^0; we take $u_h^0 = u^0$.

4. General error estimates

In this section, u is the solution of the exact problem (2.1), u_h is the solution of the discrete problem (3.1) and v_h is an arbitrary function in the finite dimensional space V_h. We estimate the error $u - u_h$ in terms of $u - v_h$ for arbitrary $v_h \in V_h$. In later applications, we will choose v_h equal to a certain interpolating function of u (see Section 5); thus the problem of estimating $u - u_h$ is reduced to the problem of estimating the interpolation error.

We will establish the following result.

Theorem 4.1

Let u be the solution of problem (2.1) and u_h be the solution of the discrete problem (3.1). Then,

$$\|\mathrm{grad}(u-u_h)\|_{G(0,t^n)} + \frac{1}{\sqrt{2}}|u^n - u_h^n|_{\Omega^n} \leq$$

$$\leq \sqrt{2}\, d \left(\sum_{\nu=0}^{n-1} \left\|\frac{\partial}{\partial t}(u-v_h)\right\|^2_{G^\nu}\right)^{1/2} + \sqrt{2}\,\|\mathrm{grad}(u-v_h)\|_{G(0,t^n)} + \tag{4.1}$$

$$+ 2 \max_{1 \leq \nu \leq n} |u^\nu - v_h^\nu|_{\Omega^\nu} + 2\sum_{\nu=1}^{n-1} |v_h^{\nu+0} - v_h^\nu|_{\Omega^\nu} ,$$

for all functions $v_h \in V_h$ and for all n, $1 \leq n \leq N$.

For the proof of this theorem we will need the following lemma.

Lemma 4.1

Let a^n and b^n, $1 \leq n \leq N$, be two sequences of non-negative real numbers which satisfy

$$(a^n)^2 + (b^n)^2 \leq \alpha a^n + \beta b^n + \sum_{\nu=1}^{n-1} \gamma^\nu b^\nu, \quad \text{for all } n , \tag{4.2}$$

where α, β and γ^ν for $1 \leq \nu \leq N-1$ are non-negative real numbers. Then

$$a^n + b^n \leq \sqrt{2}\left(\alpha + \beta + \sum_{\nu=1}^{n-1} \gamma^\nu\right) . \tag{4.3}$$

Proof of Lemma 4.1

Let $c^n = ((a^n)^2 + (b^n)^2)^{1/2}$. Then, (4.2) yields

$$(c^n)^2 \leq (\alpha+\beta)c^n + \sum_{\nu=1}^{n-1} \gamma^\nu c^\nu, \quad \text{for} \quad 1 \leq n \leq N \ .$$

Let d^n, $1 \leq n \leq N$, be the sequence of non-negative real numbers which satisfies

$$(d^n)^2 = (\alpha+\beta)d^n + \sum_{\nu=1}^{n-1} \gamma^\nu d^\nu, \quad \text{for} \quad 1 \leq n \leq N \ . \tag{4.4}$$

We have $c^1 \leq d^1 = \alpha + \beta$ and $d^n \geq \alpha + \beta$ for all n . By mathematical induction we prove that $c^n \leq d^n$ for all n: assume $c^\nu \leq d^\nu$ for $\nu = 1,2,\ldots,n-1$; then $g(c^n) \leq g(d^n)$ with $g(y) = y^2 - (\alpha+\beta)y$; since $g(y)$ is increasing for $y > \alpha + \beta$, we deduce $c^n \leq d^n$. On the other hand, we have $g(d^n) \leq g(d^{n+1})$ for all n, therefore $d^n \leq d^{n+1}$ for all n and (4.4) yields after replacing d^ν by d^n :

$$d^n \leq \alpha + \beta + \sum_{\nu=1}^{n-1} \gamma^\nu \ .$$

Finally, we have $a^n + b^n \leq \sqrt{2}\, c^n \leq \sqrt{2}\, d^n$, which ends the proof of the lemma.

Proof of Theorem 4.1

Taking $G = G^n$ and $\varphi = \varphi_h$ in (2.3) and subtracting from (3.1), we get

$$\mathcal{B}_{G^n}(u - u_h, \varphi_h) = 0, \quad \forall \varphi_h \in \Phi_h^n \ .$$

Hence, we have

$$\mathcal{B}_{G^n}(u-u_h, u-q^{(n)}u_h) = \mathcal{B}_{G^n}(u-u_h, u-q^{(n)}v_h) \ , \tag{4.5}$$

for all $v_h \in V_h$.

Let $\mathcal{B}^n(u-u_h,\ u-v_h) = \mathcal{B}_{G^n}(u-u_h,\ u-q^{(n)}v_h)$, which is an extension of the definition (3.2). The formulae (3.3) and (3.4) of Lemma 3.1 are also valid for $\mathcal{B}^n(u-u_h, u-v_h)$ with $u^{n+0} = u^n = u(\cdot, t^n)$. Hence, (4.5) yields

$$\|grad(u-u_h)\|^2_{G^n} + \frac{1}{2}|u^{n+1} - u_h^{n+1}|^2_{\Omega^{n+1}} - \frac{1}{2}|u^n - u_h^n|^2_{\Omega^n} +$$

$$+ \frac{1}{2}|u_h^{n+0} - u_h^n|^2_{\Omega^n} =$$

$$= -((u-u_h, \frac{\partial}{\partial t}(u-v_h)))_{G^n} + ((grad(u-u_h), grad(u-v_h)))_{G^n} +$$

$$+ (u^{n+1} - u_h^{n+1}, u^{n+1} - v_h^{n+1})_{\Omega^{n+1}} - (u^n - u_h^n, u^n - v_h^n)_{\Omega^n} +$$

$$+ (u^n - u_h^n, v_h^{n+0} - v_h^n)_{\Omega^n}, \quad \text{for } 0 \le n \le N-1 .$$

By summation and by application of Poincaré's inequality, we deduce:

$$\|grad(u-u_h)\|^2_{G(0,t^n)} + \frac{1}{2}|u^n - u_h^n|^2_{\Omega^n} + \frac{1}{2}\sum_{\nu=0}^{n-1} |u_h^{\nu+0} - u_h^\nu|^2_{\Omega^\nu} \le$$

$$\le \|grad(u-u_h)\|_{G(0,t^n)} \left\{ d(\sum_{\nu=0}^{n-1} \|\frac{\partial}{\partial t}(u-v_h)\|^2_{G^\nu})^{1/2} + \|grad(u-v_h)\|_{G(0,t^n)} \right\} + \tag{4.6}$$

$$+ |u^n - u_h^n|_{\Omega^n} |u^n - v_h^n|_{\Omega^n} + \sum_{\nu=1}^{n-1} |u^\nu - u_h^\nu|_{\Omega^\nu} |v_h^{\nu+0} - v_h^\nu|_{\Omega^\nu} ,$$

for $1 \le n \le N$.

Now, we will make use of Lemma 4.1. Let

$$a^n = \|grad(u-u_h)\|_{G(0,t^n)}, \quad b^n = \frac{1}{\sqrt{2}}|u^n - u_h^n| ,$$

$$\alpha^n = d(\sum_{\nu=0}^{n-1} \|\frac{\partial}{\partial t}(u-v_h)\|^2_{G^\nu})^{1/2} + \|grad(u-v_h)\|_{G(0,t^n)} ,$$

$$\beta^n = \sqrt{2} \max_{1 \le \nu \le n} |u^\nu - v_h^\nu|_{\Omega^\nu} \quad \text{and} \quad \gamma^n = \sqrt{2}\, |v_h^{n+0} - v_h^n|_{\Omega^n} .$$

Then, (4.6) yields

$$(a^n)^2 \le \alpha^n a^n + \beta^n b^n + \sum_{\nu=1}^{n-1} \gamma^\nu b^\nu, \quad \text{for } 1 \le n \le N .$$

Since the sequences $\{\alpha^n\}$ and $\{\beta^n\}$ are increasing, we deduce:

$$(a^n)^2 + (b^n)^2 \le \alpha^\ell a^n + \beta^\ell b^n + \sum_{\nu=1}^{n-1} \gamma^\nu b^\nu, \quad \text{for } 1 \le n \le \ell \le N .$$

Applying Lemma (4.1) we get:

$$a^n + b^n \leq \sqrt{2}\,(\alpha^\ell + \beta^\ell + \sum_{\nu=1}^{n-1} \gamma^\nu), \quad \text{for} \quad 1 \leq n \leq \ell .$$

The estimate (4.1) of Theorem 4.1 follows by taking $n = \ell$.

Let us notice that formula (4.5) also gives an estimate for the expression $\sum_{\nu=0}^{n-1} |u_h^{\nu+0} - u_h^\nu|^2_{\Omega^\nu}$ which involves the jumps of the function u_h .

The following theorem is a variant of Theorem 4.1.

Theorem 4.2

Assume that $V_h \cap C^0(\mathcal{S}_T)$ is non empty (where $C^0(\mathcal{S}_T)$ is the space of all continuous functions defined on $\mathcal{S}_T$) . Then, we have the estimate

$$\|\operatorname{grad}(u-u_h)\|_{G(0,t^n)} + \frac{1}{\sqrt{2}} |u^n - u_h^n|_{\Omega^n} \leq$$

(4.7) $$\leq \sqrt{2}\, d\, \left\|\frac{\partial}{\partial t}(u-v_h)\right\|_{G(0,t^n)} + \sqrt{2}\, \|\operatorname{grad}(u-v_h)\|_{G(0,t^n)} +$$

$$+ 2|u^n - v_h^n|_{\Omega^n} ,$$

for all functions $v_h \in V_h \cap C^0(\mathcal{S}_T)$ and for all n, $1 \leq n \leq N$.

Proof: In this case we have $|v_h^{\nu+0} - v_h^\nu| = 0$. Then, (4.6) yields

$$(a^n)^2 + (b^n)^2 \leq \alpha^n a^n + \tilde{\beta}^n b^n, \quad \text{where } a^n,\ b^n \text{ and } \alpha^n$$

are defined as in the proof of Theorem 4.1 and $\tilde{\beta} = \sqrt{2}\,|u^n - v_h^n|_{\Omega^n}$. It follows $a^n + b^n \leq \sqrt{2}(\alpha^n + \tilde{\beta}^n)$. ■

Notation: We introduce now certain general notations which are needed in the following section.

$x_1, x_2, \ldots, x_m$ = space-coordinates,

$j = (j_0, j_1, \ldots, j_m)$ = multi-index with $j_0, j_1, \ldots, j_m \geq 0$,

$j' = (j_1, \ldots, j_m)$,

$|j| = \sum_{\ell=0}^{m} |j_\ell|, \quad |j'| = \sum_{\ell=1}^{m} |j_\ell|$,

$$\partial^j u = \partial^{|j|} u / \partial t^{j_0} \, \partial x_1^{j_1} \ldots \partial x_m^{j_m} ,$$

$$\partial^{j'} u = \partial^{|j'|} u / \partial x_1^{j_1} \ldots \partial x_m^{j_m} ,$$

$$\| D^s u \|_G = \Big(\sum_{|j|=s} \| \partial^j u \|_G^2 \Big)^{1/2}, \quad \text{for any integer } s \geq 0 \text{ and any } G \subset \mathcal{D}_T$$

$$|D^s u|_{\Omega(t)} = \Big(\sum_{|j'|=s} |\partial^{j'} u(\cdot,t)|^2_{\Omega(t)} \Big)^{1/2}, \quad \text{for any } t, \; 0 \leq t \leq T .$$

The two foregoing expressions are defined provided

$$u \in H^s(\mathcal{D}_T) \cap C^0([0,T]; \; H^s(\Omega(t))) .$$

5. Space-time finite elements

In this section, we make a specific choice for the spaces Φ_h^n; we use space-time finite elements as in [3], [4], [11], [12]. For simplicity, we assume that the domain $\mathcal{O}_T$ is polyhedral; if not, we should use curved finite elements near the boundary.

5.a) Simplicial elements

Let h be a "small" positive number and for each n, $0 \le n \le N-1$, let $\mathcal{T}_h^n$ be a finite set of $(m+1)$-simplices K which satisfy the conditions

(5.1) $\bar{G}^n = \{\cup K;\ K \in \mathcal{T}_h^n\}$,

(5.2) $h(K) \le h$, for all $K \in \mathcal{T}_h^n$,

where $h(K)$ denotes the diameter of K ,

(5.3) $\mathring{K} \cap \mathring{K}' = \phi$, $\forall\, K, K' \in \mathcal{T}_h^n$, where $\mathring{K}$ denotes the interior of K ,

(5.4) If a vertex of K belongs to K', then it is also a vertex of K', for all $K, K' \in \mathcal{T}_h^n$.

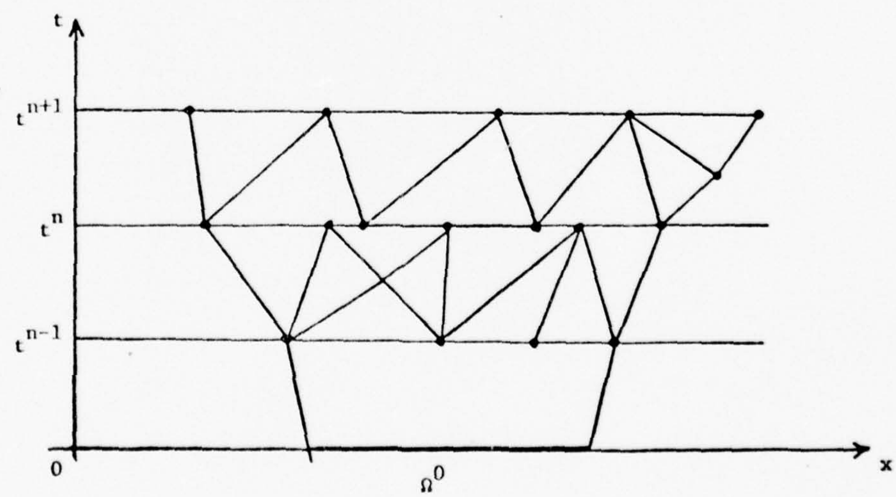

Figure 2: A triangulation of the subdomains G^{n-1} and G^n in the one-dimensional case.

We will denote $\mathcal{T}_h = \{ \cup \mathcal{T}_h^n ;\ 0 \leq n \leq N-1 \}$. Let k be an integer, $k \geq 1$; let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$ with respect to the variables $t, x_1, \ldots, x_m$ and let $\mathcal{P}_k(K)$ be the set of their restrictions to K. For each n, we choose Φ_h^n equal to the space of all continuous functions defined on $\bar{G}^n$ which vanish on the lateral boundary of G^n and whose restriction to each element K coincides with a polynomial of degree $\leq k$, i.e.

$$\Phi_h^n = \{\varphi_h ;\ \varphi_h \in \Phi(G^n),\ \varphi_h|_K \in \mathcal{P}_k(K),\ \forall K \in \mathcal{T}_h^n\} . \tag{5.5}$$

For the statement of the next theorem, we need the following definition. Let E_K denote the set of all the edges of a $(m+1)$-simplex K, with $m+1 = p \geq 2$, and let $\theta(D,D')$ denote the angle of two arbitrary straight lines D and D' in $\mathbb{R}^p$.

Definition 5.1

The condition angle of a p-simplex K is the angle

$$\theta(K) = \max_{D \in \mathbb{R}^p} \min_{D' \in E_K} \theta(D,D') . \tag{5.6}$$

Remarks: For all non-degenerate p-simplices K, we have $0 < \theta < \pi/2$. The condition angle $\theta(K)$ approaches $\pi/2$ if and only if all the edges of K are almost parallel to a same hyperplane. In the case $p = 2$, K is a triangle and $\theta(K)$ is equal to one half of the largest angle of K.

Theorem 5.1

Let the space Φ_h^n be chosen according to (5.5). Let u be the solution of problem (2.1) and u_h be the solution of problem (3.1). Assume $u \in H^\ell(\mathcal{S}_T) \cap C^0([0,T]; H^\ell(\Omega(t)))$ where ℓ is the smallest integer such that $\ell \geq k+1$ and $\ell > (m+1)/2$. Assume that the discretization of $\mathcal{S}_T$ satisfies the conditions

$$h \leq 1, \quad Nh \leq \lambda , \tag{5.7}$$

$$\theta(K) \leq \theta_0 < \pi/2, \quad \text{for all } K \in \mathcal{T}_h , \tag{5.8}$$

where λ and θ_0 are given positive constants. Then,

$$\|\operatorname{grad}(u_h - u)\|_{G(0,t^n)} + |u_h^n - u^n|_{\Omega^n} \leq \gamma h^k , \tag{5.9}$$

for $1 \le n \le N$, with

$$\gamma = C \sum_{s=k+1}^{\ell} ((\cos\theta_0)^{-1} \|D^s u\|_{G(0,t^n)} + \lambda \operatorname{Max}_{0\le t\le t^n} |D^s u(\cdot,t)|_{\Omega(t)}) ,$$

where C is a positive constant which depends only on m, k and d. Moreover, we have the following estimate for the jumps of u_h:

$$(5.10) \qquad (\sum_{n=0}^{N-1} |u_h^{n+0} - u_h^n|^2)^{1/2} \le C' \gamma h^k ,$$

where C' is an absolute constant.

Proof: For each element K, let $\mathcal{J}(K)$ be the set of the points of K whose barycentric coordinates with respect to the vertices of K are multiples of $1/k$ and let Π_K denote the interpolation operator which associates to each function $\varphi \in C^0(\bar{K})$ the unique function $\Pi_K\varphi \in \mathcal{P}_k(K)$ such that $\Pi_K\varphi = \varphi$ at all the points of $\mathcal{J}(K)$ (see [18], [5]). Let Π_h denote the interpolation operator which associates to each function $\varphi \in C^0(\bar{\mathcal{S}}_T)$ the unique function $\Pi_h\varphi \in V_h$ such that $\Pi_h\varphi = \Pi_K r_K\varphi$ in the interior of K, for all $K \in \mathcal{T}_h$, where r_K denotes the operator of restriction to K. Note that the interpolated function $\Pi_h\varphi$ is in general discontinuous at the time $t = t^n$ since the sets $\mathcal{T}_h^{n-1}$ and $\mathcal{T}_h^n$ are independent.

We will use the estimate of Theorem 4.1 with $v_h = \Pi_h u$ (note that $\Pi_h u$ is defined since the hypotheses of the present theorem together with Sobolev's imbedding theorem imply that u is continuous on $\bar{\mathcal{S}}_T$). Let $\mathcal{T}'_{h,n}$ (resp. $\mathcal{T}'_{h,n+0}$) denote the set of all m-simplices K' which lie in the hyperplane $t = t^n$ and which are a face of an element $K \in \mathcal{T}_h^{n-1}$ (resp. $\mathcal{T}_h^n$). Using the inequality

$$|v_h^{\nu+0} - v_h^{\nu}| \le |v_h^{\nu+0} - u^{\nu}| + |v_h^{\nu} - u^{\nu}| ,$$

we deduce from (4.1):

$$\|\operatorname{grad}(u-u_h)\|_{\mathcal{S}_T} + \frac{1}{\sqrt{2}} |u-u_h|_{\Omega^N} \le$$

$$(5.11) \qquad \le \sqrt{2}\, d (\sum_{K\in\mathcal{T}_h} \|\frac{\partial}{\partial t}(u-\Pi_h u)\|_K^2)^{1/2} +$$

$$+ \sqrt{2} (\sum_{K\in\mathcal{T}_h} \|\operatorname{grad}(u - \Pi_h u)\|_K^2)^{1/2} +$$

$$+ 2N \max_{1 \le \nu \le N} \Big(\sum_{K' \in \mathfrak{T}'_{h,\nu}} |u - \Pi_h u|^2_{K'} \Big)^{1/2} +$$

$$+ 2(N-1) \max_{1 \le \nu \le N-1} \Big(\sum_{K' \in \mathfrak{T}'_{h,\nu+0}} |u - \Pi_h u|^2_{K'} \Big)^{1/2} ,$$

with $\Pi_h u = (\Pi_h u)^{\nu+0}$ on each $K' \in \mathfrak{T}'_{h,\nu+0}$.

Theorem 2.3 of [10] (*) yields:

(5.12) $$\Big\| \frac{\partial}{\partial t}(u - \Pi_h u) \Big\|_K + \| \mathrm{grad}(u - \Pi_h u) \|_K \le \frac{C_1}{\cos \theta_0} h^k \sum_{s=k+1}^{\ell} \| D^s u \|_K ,$$

for $h \le 1$ and for all $K \in \mathfrak{T}_h$,

(5.13) $$|u - \Pi_h u|_{K'} \le C_2 h^{k+1} \sum_{s=k+1}^{\ell} |D^s u|_{K'} ,$$

for $h \le 1$ and for all $K' \in \mathfrak{T}'_{h,\nu} \cup \mathfrak{T}'_{h,\nu+0}$,

where C_1 and C_2 are constants which depend only on k and m. The estimate (5.9) follows at once from (5.11), (5.12) and (5.13), where N can be replaced by an arbitrary value of n. The estimate (5.10) is obtained by using (5.9) in the right hand side member of (4.6). ∎

Remark 5.1: The condition (5.8) which was established in [10] is an improvement on the classical condition of Zlamal [25] and Ciarlet-Raviart [5]. It is valid for a general class of interpolation operators which contains the operator Π_K considered above and for general error estimates in the Sobolev spaces $W^{\mu,q}$, μ integer ≥ 0, $1 \le q \le \infty$. The same condition has been established independently from the author by Babuška and Aziz [1] in the particular case $m = 1$ (the elements K are triangles), $k = 1$ or 2, $\mu = 1$ and $q = 2$.

Remark 5.2: Suppose that the triangulation $\mathfrak{T}_h$ satisfies the condition

(5.14) $$\mathfrak{T}'_{h,n} = \mathfrak{T}'_{h,n+0} , \text{ for } 1 \le n \le N-1 .$$

Then, $\Pi_h u \in C^0(\bar{\mathcal{Q}}_T)$ and we can apply Theorem 4.2. We get the same estimate as in Theorem

(*) There is a typographical error in the statement of this theorem: one should read $W^{k-m+\ell,p}$ and $W^{k+\ell,p}$ instead of $W^{k-m+1,p}$ and $W^{k+1,p}$.

5.1, but without the hypothesis $Nh \leq \lambda$; in the expression of γ, the constant λ must be replaced by 1 (or by h).

Remark 5.3: The condition $Nh \leq \lambda$ implies that the average value of the time-steps should not be too small; it should be at least of the same order as h. In particular, we can take $N = 1$, which yields a completely implicit method for the whole domain $\mathcal{J}_T$. The advantage of dividing the interval $[0,T]$ into sub-intervals is to split the global system of discrete equations (3.1) into subsystems each of which corresponds to one sub-domain G^n.

5.b) Prismatic elements

For simplicity, we will use the vocabulary corresponding to the case $m = 2$; thus, we will say "triangle" instead of "m-simplex", "plane" instead of "hyper-plane", "prism" instead of "hyper-prism", ...

For each n, $0 \leq n \leq N-1$, let $\mathcal{T}_h^n$ be a finite set of elements $K = K_i^n$, $1 \leq i \leq I$, which satisfy (5.1), (5.2), (5.3) and the following properties: the section of each element K_i^n by the plane $t = t^n$ (resp. $t = t^{n+1}$) is a triangle T_i^{n+0} (resp. T_i^{n+1}) and the section of K_i^n by any plane t=constant, $t^n \leq t \leq t^{n+1}$, is a triangle whose vertices are located on three straight segments $[P_\ell^{n+0}\ P_\ell^{n+1}]$ where P_ℓ^{n+0} is a vertex of T_i^{n+0} and P_ℓ^{n+1} is a vertex of T_i^{n+1}; the element K_i^n is a distorted prism (see Figure 3); in general, it is not a polyhedral; the triangles T_i^{n+0} and T_i^{n+1} are called the bases of the element K_i^n and their vertices are called the vertices of K_i^n. All the vertices of the elements $K \in \mathcal{T}_h^n$ must satisfy condition (5.4). The sets $\mathcal{T}_h^n$ and $\mathcal{T}_h^{n+1}$ are independent; in particular, the set $\mathcal{T}'_{h,n+0}$ of all triangles T_i^{n+0} which are a base of one element $K \in \mathcal{T}_h^n$ is independent of the set $\mathcal{T}'_{h,n}$ of all triangles T_i^{n+1} which are a base of one element $K \in \mathcal{T}_h^{n-1}$ (this is a difference with the continuous finite element method used in [4]; in that method we imposed $\mathcal{T}'_{h,n+0} = \mathcal{T}'_{h,n}$) (See figure 4).

Let k' and k'' be two integers, $k' \geq 1$, $k'' \geq 1$. For each element $K \in \mathcal{T}_h^n$, let $Q_{k',k''}(K)$ be the space of all functions defined in K whose restriction to each section of K by a plane t=constant is a polynomial of degree k' with respect to the space variables and whose restriction to each edge $[P_\ell^{n+0}\ P_\ell^{n+1}]$ is a polynomial of degree k'' with respect to t. We take

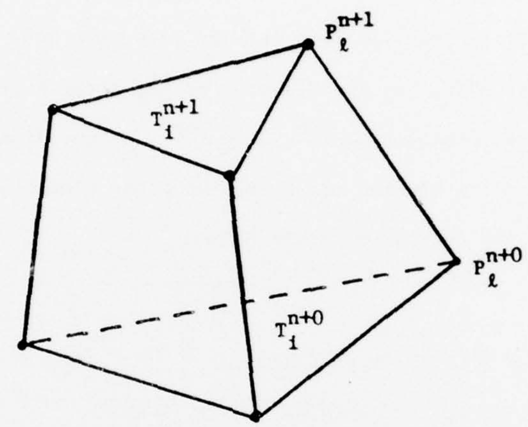

Figure 3: A "prismatic" element K_i^n (two-dimensional case)

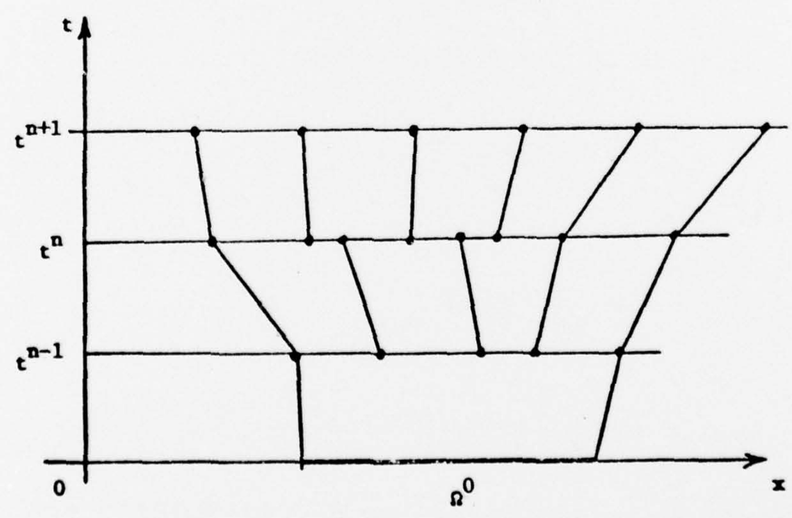

Figure 4: Trapezoidal elements (one-dimensional case)

$$\Phi_h^n = \{\varphi_h;\ \varphi_h \in \Phi(G^n),\ \ r_K\varphi_h \in Q_{k',k''}(K),\ \ K \in \mathcal{T}_h^n\}\ , \tag{5.15}$$

where r_K denotes the restriction to K .

Let $k = \min\{k',k''\}$. [illegible] ℓ be the smallest integer such that $\ell \geq k + 1$ and $\ell > (m+1)/2$. Let ℓ' be the smallest integer such that $\ell' \geq k' + 1$ and $\ell' > m/2$. For each $K \in \mathcal{T}_h^n$, let $h(K)$ be the diameter of K, $h'(K) = (t^{n+1} - t^n)$ be the "height" of K and $\rho(K)$ be the minimum for $t^n \leq t \leq t^{n+1}$ of the diameter of the largest circle contained in the triangular section of K by the plane t=constant. Let $\mathcal{T}_h = \cup\{\mathcal{T}_h^n;\ 0 \leq n \leq N-1\}$. Then, the following result holds:

Theorem 5.2

Let the space Φ_h^n *be chosen according to* (5.15). *Let* u *be the solution of problem* (2.1) *and* u_h *be the solution of problem* (3.1). *Assume* $u \in H^{\ell}(\mathcal{O}_T) \cap C^0([0,T];\ H^{\ell'}(\Omega(t)))$. *Assume*

$$Nh \leq \lambda\ , \tag{5.16}$$

$$h(K) \leq h \leq 1\ , \tag{5.17}$$

$$h'(K)/h(K) \geq \sigma_0\ , \tag{5.18}$$

$$\rho(K)/h(K) \geq \sigma_1,\ \text{ for all } K \in \mathcal{T}_h\ , \tag{5.19}$$

where λ, σ_0 *and* σ_1 *are positive constants. Then,*

$$\|\operatorname{grad}(u_h-u)\|_{G(0,t^n)} + |u_h^n - u^n|_{\Omega^n} < \gamma h^k + \gamma' h^{k'}\ , \tag{5.20}$$

for $1 \leq n \leq N$, *with*

$$\gamma = C \sum_{s=k+1}^{\ell} \|D^s u\|_{G(0,t^n)} \qquad \textit{and}$$

$$\gamma' = \lambda C' \sum_{s=k'+1}^{\ell'} \operatorname{Max}\{|D^s u(\cdot,t)|_{\Omega(t)};\ \ 0 \leq t \leq t^n\}\ ,$$

where C *and* C' *are two positive constants which depend only on* m, k', k'', σ_0, σ_1 *and* d . *Moreover*

$$\left(\sum_{n=0}^{N-1} |u_h^{n+0} - u_h^n|^2\right)^{1/2} \leq C_0(\gamma h^k + \gamma' h^{k'})\ , \tag{5.21}$$

where C_0 is an absolute constant.

For the proof of this theorem, we need a preliminary result concerning the interpolation error in each of the elements K.

For each $K \in \mathcal{T}_h^n$, let $\mathcal{J}(K)$ be the set of the points of K which are located in one of the planes $t = t^n + \frac{j}{k''}(t^{n+1} - t^n)$ with j integer, $0 \leq j \leq k''$, and whose barycentric coordinates with respect to the vertices of the corresponding triangular section of K are multiples of $1/k'$. Let Π_K be the interpolation operator which associates to each function $\varphi \in C^0(\bar{K})$ the unique function $\Pi_K\varphi \in Q_{k',k''}(K)$ such that $\Pi_K\varphi = \varphi$ at all the points of $\mathcal{J}(K)$. Then, the following result holds.

Lemma 5.1

Suppose that K satisfies conditions (5.17), (5.18) and (5.19). Let ℓ be the smallest integer such that $\ell \geq k + 1$ and $\ell > (m+1)/2$. Then, for all functions $\varphi \in H^\ell(K)$, we have

$$\|D(\varphi - \Pi_K\varphi)\|_K \leq C_1 h^k \sum_{s=k+1}^{\ell} \|D^s\varphi\|_K , \tag{5.22}$$

where C_1 is a constant which depends on m, k', k'', σ_0 and σ_1.

Proof of Lemma 5.1

Let $\mathcal{P}_k(K)$ be the space of the restrictions to K of all polynomials of degree $\leq k$ with respect to the variables $t, x_1, \ldots, x_m$. The space $\mathcal{P}_k(K)$ is invariant under the interpolation operator Π_K. Hence, applying Lemma 2.5 of [10] which is a variant of Lemma 7 of Ciarlet-Raviart [5], we get

$$\|D(\varphi - \Pi_K\varphi)\|_K \leq C_1(K) \sum_{s=k+1}^{\ell} \|D^s\varphi\|_K , \forall\varphi \in H^\ell(K) , \tag{5.23}$$

where $C_1(K)$ is a constant which depends on the element K and on the operator Π_K.

Let us assume $h(K) = 1$; then, we have $\rho(K) \geq \sigma_0$ and $h'(K) \geq \sigma_1$. The set $\mathcal{K}$ of all the elements K which satisfy these conditions is compact. Taking the maximum of $C_1(K)$ for all $K \in \mathcal{K}$ we can replace $C_1(K)$ in (5.23) by a constant C_1 which depends only on m, k', k'', σ_0 and σ_1. The estimate (5.22) for $h \leq 1$ follows by a simple change of scale.

Proof of Theorem 5.2

Let Π_h be the interpolation operator which associates to each function $\varphi \in C^0(\mathcal{T}_T)$ the function $\Pi_h \varphi \in V_h$ such that $\Pi_h \varphi = \Pi_K r_K \varphi$ in the interior of K, for all $K \in \mathcal{T}_h$. Theorem 4.1 with $v_h = \Pi_h u$ yields (5.11) as in the proof of Theorem 5.1. The estimate (5.20) follows by application of Lemma 5.1, of the estimate (5.13) with k' instead of k and of condition (5.16). Then, the estimate (5.21) follows from (4.6).

6. An example

In this section, we give an explicit expression of the discrete equations in a simple case: uniform rectangular grid, approximation of degree 1. Thus we get a finite difference scheme which is a particular case of our method. We will assume $f = 0$.

Let K_{i-1}^{n} and K_{i}^{n} be two neighbor rectangular elements with vertices P_{i-1}^{n}, P_{i-1}^{n+1}, P_{i}^{n}, P_{i}^{n+1}, P_{i+1}^{n}, P_{i+1}^{n+1} where $P_{i}^{n} = (x_{i}, t^{n})$, $x_{i+1} - x_{i} = x_{i} - x_{i-1} = \Delta x$, $t^{n+1} - t^{n} = \Delta t$. Let $u_{i}^{n} = u_{h}(P_{i}^{n}) = u_{h}^{n}(x_{i})$ and $u_{i}^{n+0} = u_{h}^{n+0}(x_{i}) = \lim \{ u_{h}(x_{i}, t^{n} + \varepsilon);\ \varepsilon > 0,\ \varepsilon \to 0 \}$. In each rectangle K_{i}^{n}, u_{h} is linear with respect to each variable x and t separately and uniquely determined by the four values u_{i}^{n+0}, u_{i+1}^{n+0}, u_{i}^{n+1} and u_{i+1}^{n+1} by means of the formula

(6.1) $$u_{h}(x,t) = (1 - \hat{x})(1 - \hat{t})\, u_{i}^{n+0} + \hat{x}(1 - \hat{t})\, u_{i+1}^{n+0} + (1 - \hat{x})\hat{t}\, u_{i}^{u+1} + \hat{x}\, \hat{t}\, u_{i+1}^{n+1} ,$$

where $\hat{x} = (x - x_{i})/ \Delta x$ and $t = (t - t^{n})/\Delta t$.

At each time step, the values u_{i}^{n} are known for all i and we must determine the values u_{i}^{n+0} and u_{i}^{n+1} . The corresponding equations are obtained by writing the integral relation (3.1) for linearly independent test-functions φ_{h}. For example, we can take, for each i, the two test functions $\varphi_{h}^{(i,1)}$ and $\varphi_{h}^{(i,2)}$ such that

$$\varphi_{h}^{(i,1)}(P) = \begin{cases} 1 & \text{if } P = P_{i}^{n} \\ -1 & \text{if } P = P_{i}^{n+1} \\ 0 & \text{at all other grid-points,} \end{cases}$$

$$\varphi_{h}^{(i,2)}(P) = \begin{cases} 1 & \text{if } P = P_{i}^{n+1} \\ 0 & \text{at all grid-points } P \neq P_{i}^{n+1} \end{cases} .$$

Thus, we get the following equations.

(6.2) $$\left(\frac{1}{6}(u_{i+1}^{n+0} - u_{i+1}^{n}) + \frac{2}{3}(u_{i}^{n+0} - u_{i}^{n}) + \frac{1}{6}(u_{i-1}^{n+0} - u_{i-1}^{n})\right)/\Delta t +$$

$$+ \frac{1}{6}\left((u_{i+1}^{n+1} - 2\, u_{i}^{n+1} + u_{i-1}^{n+1}) - (u_{i+1}^{n+0} - 2u_{i}^{n+0} + u_{i-1}^{n+0})\right)/(\Delta x)^{2} = 0$$

(6.3) $$\left(\frac{1}{6}(u_{i+1}^{n+1} - u_{i+1}^{n+0}) + \frac{2}{3}(u_{i}^{n+1} - u_{i}^{n+0}) + \frac{1}{6}(u_{i-1}^{n+1} - u_{i-1}^{n+0})\right)/\Delta t -$$

$$- \left(\frac{1}{3}(u_{i+1}^{n+0} - 2u_{i}^{n+0} + u_{i-1}^{n+0}) + \frac{2}{3}\,(u_{i+1}^{n+1} - 2u_{i}^{n+1} + u_{i-1}^{n+1})\right)/(\Delta x)^{2} = 0 .$$

7. General parabolic problems

In this section, we extend the numerical method described in Section 3 to more general parabolic problems. Some new notations are introduced; the others are the same as in the preceding sections.

7.a) The continuous problem

We consider problems of the form

$$
(7.1)\quad
\begin{aligned}
&\text{a)}\quad \frac{\partial u}{\partial t} - Au = f \quad \text{in } \mathcal{Q}_T \\
&\text{b)}\quad Bu = 0 \qquad\quad\ \text{on } \Sigma_T \\
&\text{c)}\quad u = u^0 \qquad\quad\ \text{in } \Omega(0)\ ,
\end{aligned}
$$

where A is a differential operator of order 2μ with respect to the space variables, μ positive integer, and B is a boundary operator.

We will give a variational formulation of problem (7.1) which is a generalization of the integral relation (2.3). For each $t \in [0,T]$, let $\mathcal{V}(t)$ be a subspace of the Sobolev space $H^{\mu}(\Omega(t))$ which is dense in $L^2(\Omega(t))$ and let $\mathcal{V}'(t)$ be the dual space of $\mathcal{V}(t)$. For any domain $G = G(\tau_0,\tau_1) = \{(x,t);\ x \in \Omega(t),\ \tau_0 < t < \tau_1\}$, let

$$\mathcal{E}(G) = L^2(\tau_0,\tau_1;\ \mathcal{V}(t))\ ,$$

$$\mathcal{E}'(G) = L^2(\tau_0,\tau_1;\ \mathcal{V}'(t)) = \text{dual space of } \mathcal{E}(G)\ ,$$

$$\mathcal{W}(G) = \{\varphi;\ \varphi \in \mathcal{E}(G),\ \frac{\partial\varphi}{\partial t} \in \mathcal{E}'(G)\}\ .$$

Let $\|\cdot\|_{\mathcal{E}(G)}$ denote the norm in $\mathcal{E}(G)$ defined by

$$\|\varphi\|_{\mathcal{E}(G)} = \Big(\sum_{|j'|\le\mu} \|\partial^{j'}\varphi\|_G^2\Big)^{1/2}, \quad \text{with } j' = (j_1,\dots,j_m)\ ,$$

for all $\varphi \in \mathcal{E}(G)$.

Let $((\cdot,\cdot))_G$ denote the duality between $\mathcal{E}(G)$ and $\mathcal{E}'(G)$ obtained by extension of the inner product in $L^2(G)$.

We have $\mathcal{E}(G) \subset L^2(G) \subset \mathcal{E}'(G)$ and, by a lemma of Lions [14], $\mathcal{W}(G) \subset C^0([\tau_0,\tau_1];\ L^2(\Omega(t)))$.

Let us assume $A \in \mathcal{L}(\mathcal{E}(G), \mathcal{E}'(G))$, i.e. A is a linear continuous operator from $\mathcal{E}(G)$ into $\mathcal{E}'(G)$ and let us define

(7.2) $$a_G(\psi,\varphi) = ((A\psi,\varphi))_G \text{ , for all } \psi \text{ and } \varphi \text{ in } \mathcal{E}(G) .$$

$a_G(\cdot,\cdot)$ is a bilinear form defined and continuous on $\mathcal{E}(G) \times \mathcal{E}(G)$. Assume that it satisfies the two following properties.

Uniform continuity:

(7.3) $$|a_G(\psi,\varphi)| \leq M\|\psi\|_{\mathcal{E}(G)} \|\varphi\|_{\mathcal{E}(G)} ,$$

for all ψ and φ in $\mathcal{E}(G)$ and for all G, where M is a constant independent of G .

Uniform coercivity:

(7.4) $$a_G(\varphi,\varphi) + \delta\|\varphi\|_G^2 \geq \eta\|\varphi\|^2_{\mathcal{E}(G)} ,$$

for all $\varphi \in \mathcal{E}(G)$ and for all G, where δ and η are two constants independent of G . In fact, we will consider only the reduced case $\delta = 0$ obtained by a standard change of variable $u \to u\, e^{\beta t}$, $\beta > 0$, in (7.1) .

Let

(7.5) $$\mathcal{B}_G(\psi,\varphi) = -((\psi,\frac{\partial\varphi}{\partial t}))_G + a_G(\psi,\varphi) + (\psi,\varphi)_{\Omega(\tau_1)} + \\ -(\psi,\varphi)_{\Omega(\tau_0)} .$$

$\mathcal{B}_G(\cdot,\cdot)$ is a bilinear form defined and continuous on $\mathcal{S}(G) \times \mathcal{S}(G)$.

Variational problem: Find $u \in \mathcal{S}(\mathcal{J}_T)$ such that $u(\cdot,0) = u^0$ and

(7.6) $$\mathcal{B}_G(u,\varphi) = ((f,\varphi))_G \text{ , for all } \varphi \in \mathcal{S}(G)$$

and for all $G = G(\tau_0,\tau_1) \subset \mathcal{J}_T$.

This formulation is equivalent to the formulation of Lions [14]. For the existence and uniqueness of the solution, see [14].

A standard integration by parts shows that (7.6) is a weak form of the differential equation (7.1a). As for the boundary conditions, they depend on the subspace $\mathcal{V}(t)$. We give two examples

Example 7.1

Let $\mathcal{V}(t) = H_0^{\mu}(\Omega(t))$. Then, the solution u of problem (7.6) satisfies the Dirichlet boundary conditions

$$\gamma_j u = 0 \quad \text{on} \quad \Sigma_T \ , \text{for} \ \ j = 0,1,\ldots,\mu-1 \ ,$$

where $\gamma_j u(\cdot,t) = \partial^j u(\cdot,t)/\partial\nu^j$ denotes the trace of order j of the function $u(\cdot,t)$ on $\Gamma(t)$.

Example 7.2

Suppose $\mu = 1$ and $A = \Delta$ = Laplacian operator. For each t, let $\Gamma^{(1)}(t)$ be a subset of $\Gamma(t)$ and $\mathcal{V}(t) = \{\varphi;\ \varphi \in H^1(\Omega(t)),\ \varphi = 0 \ \text{on} \ \Gamma^{(1)}(t)\}$. Let $\sum_T^{(1)} = \{\cup\, \Gamma^{(1)}(t);\ 0 < t < T\}$ and $\sum_T^{(2)} = \sum_T - \sum_T^{(1)}$. Then, the solution u of problem (7.6) satisfies (in a weak sense) the boundary conditions

$$\begin{cases} u = 0 \quad \text{on} \quad \sum_T^{(1)} \ , \\ s_\nu \, u + \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \sum_T^{(2)} \ , \end{cases}$$

where s_ν denotes the outward normal speed of propagation of the boundary $\Gamma(t)$ and $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u .

7.b) Discontinuous approximations

Our method of approximation is based on the fact that relation (7.6) makes sense even if we do not assume $\frac{\partial u}{\partial t} \in \mathcal{E}(G)$ since $\frac{\partial u}{\partial t}$ does not appear in (7.6). Therefore, this relation can be used to define approximations which are not in $\mathcal{B}(\mathcal{J}_T)$.

Let G^n be defined as in Section 3 and let Φ_h^n be, for each n, a finite dimensional subspace of $\mathcal{B}(G^n)$. Let V_h be the corresponding space defined as in Section 3; V_h is a finite dimensional subspace of $\mathcal{E}(\mathcal{J}_T)$, but $V_h \not\subset \mathcal{B}(\mathcal{J}_T)$; if $v_h \in V_h$, the function $t \to v_h(\cdot,t) \in L^2(\Omega(t))$ is in general discontinuous at the times $t = t^n$, $0 \le n \le N$. As in Section 3, we denote $v_h^n = v_h(\cdot,t^n)$ for $n = 0,1,\ldots,N$ and $v_h^{n+0} = \lim\{v_h(\cdot,t^n + \varepsilon);\ \varepsilon > 0,\ \varepsilon \to 0\}$ for $n = 0,1,\ldots,N-1$. The discrete problem is formulated exactly as in Section 3.

Discrete problem: Find $u_h \in V_h$ such that $u_h^0 = u^0$ and

(7.7) $$\mathcal{B}_{G^n}(u_h,\varphi_h) = ((f,\varphi_h))_{G^n} ,$$

for all $\varphi_h \in \Phi_h^n$ and all n, $0 \le n \le N-1$.

Now, we state two theorems which are generalizations of Theorems 3.1 and 4.1.

Theorem 7.1 (existence, uniqueness, stability)

The discrete problem (7.7) admits a unique solution u_h which satisfies the estimate

(7.8) $$\eta \| u_h \|^2_{\mathcal{E}(G(0,t^n))} + \frac{1}{2} | u_h^n |^2_{\Omega^n} \le$$

$$\le \left(\frac{1}{\sqrt{2}} |u^0|_{\Omega^0} + \frac{1}{\sqrt{\eta}} \| f \|_{G(0,t^n)} \right)^2 ,$$

for all n, $0 \le n \le N$.

Theorem 7.2 (error estimate)

Let u be the solution of problem (7.6) and u_h be the solution of the discrete problem (7.7). Then

$$\sqrt{\eta} \, \| u-u_h \|_{\mathcal{E}(G(0,t^n))} + \frac{1}{\sqrt{2}} |u^n-u_h^n|_{\Omega^n} \le$$

(7.9) $$\le (2/\eta)^{1/2} \left(\sum_{\nu=0}^{n-1} \| \frac{\partial}{\partial t}(u-v_h) \|^2_{G^\nu} \right)^{1/2} +$$

$$+ (2/\eta)^{1/2} M \|u-v_h\|_{\mathcal{E}(G(0,t^n))} + 2 \max_{1\le\nu\le n} |u^\nu-v^\nu|_{\Omega^\nu} +$$

$$+ 2 \sum_{\nu=1}^{n-1} |v_h^{\nu+0} - v_h^\nu|_{\Omega^\nu} ,$$

for all functions $v_h \in V_h$ and all n, $1 \le n \le N$.

Proof of Theorem 7.1

Same argument as for Theorem 3.1. We use the following inequality which is a generalization of (3.4) in Lemma 3.1.

$$\mathcal{B}^n(v_h,v_h) \ge \alpha \|v_h\|^2_{\mathcal{E}(G^n)} + \frac{1}{2}|v_h^{n+1}|^2_{\Omega^{n+1}} - \frac{1}{2}|v_h^n|^2_{\Omega^n} + \frac{1}{2}|v_h^{n+0} - v_h^n|^2_{\Omega^n} ,$$

with $\mathcal{B}^n(v_h, v_h) = \mathcal{E}_{G^n}(v_h, q^{(n)} v_h)$.

We also use the obvious inequality $\|v_h\|_G \leq \|v_h\|_{\mathcal{E}(G)}$.

Proof of Theorem 7.2

Same argument as for Theorem 3.2.

Applications to finite elements

As in Section 5, the finite dimensional spaces Φ_h^n can be defined by means of space-time finite elements. Then, the function v_h in the right hand side member of (7.9) can be replaced by an interpolate of the solution u and the interpolation error can be estimated by using standard results of approximation theory.

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~~We consider~~ general linear parabolic equations are considered in a given time dependent domain and ~~we describe~~ a general class of Galerkin-type approximations is considered which are continuous with respect to the space variables, but which admit discontinuities with respect to time at each time step. Unconditional stability is proved and a general error estimate is established. These results are applied to certain finite element methods based on space-time finite elements.

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